Fractal Model of the Atom in the Hydrodynamic Approach of Scale Relativity Theory

ADINA ANTICI[1], VIOREL-PUIU PAUN[2]*, PETRU NICA[1] , MARICEL AGOP[1]

[1] " Gh. Asachi" Technical University Iasi, Department of Physics, 71 Mangeron Blvd.,700050, Iasi, Romania

[2] Politehnica University of Bucharest, Faculty of Applied Sciences, Department of Physics I, 313 Splaiul Independentei, 060042, Bucharest, Romania

In this paper the fractal model of the atom, using the hydrodynamic approach of the scale relativity theory, is obtained. Thus, assuming that the electron motion around the nucleus takes place on fractal curves of fractal dimension D_F (continuous but non-differentiable curves), it is shown that its dynamics, in the second order approximation of the equation of motion, is described in complex speed field by a generalized Navier-Stokes type equation with imaginary viscosity coefficient. Applying this model to study the atom, it resulted that the real part of the complex velocity field describes the electron averaged movement. The electron moves on stationary orbits according to a quantification condition and the imaginary part of the complex velocity describes the fractality through a fractal potential. In the D_F=2 fractal dimension and for the $D=\hbar/2m$ viscosity coefficient, the classical results of quantum mechanics for the hydrogen atom are obtained.

Keywords: fractal, scale theory, hydrodynamics, energy quantification, stationary orbits

The theoretical description of microphysical systems is generally based on Schrödinger's wave mechanics [1,2], Heisenberg's matrix mechanics [3], or on Feynman's path-integral mechanics [4]. Another approach is the hydrodynamic formulation of quantum mechanics belonging to Madelung, De Broglie, Takabayasi and Bohm (idea of "subquantum medium") [5]. The hydrodynamic theory of quantum mechanics has been later extended by De Broglie (idea of the "double solution") and used as preliminary theoretical scheme for quasi-causal interpretations of microphysical phenomena [5,6].

The scale relativity theory (SRT) is a new approach to understand quantum mechanics, and moreover physical domains involving scale laws, such as chaotic systems [7,8]. It is based on a generalization of Einstein's principle of relativity to scale transformations. Namely, one redefines space-time resolutions as characterizing the state of scale of reference systems, in the same way as velocity characterizes their state of motion. Then one requires that the laws of physics apply whatever the state of the reference system, of motion (principle of motion-relativity) and of scale (principle of scale-relativity). The principle of scale-relativity is mathematically achieved by the principle of scale-covariance, requiring that the equations of physics keep their simplest form under transformations of resolution [7,8].

It is well known that the geometrical tool that implements Einstein's general motion-relativity is the concept of Riemannian, curved space-time. In a similar way, the concept of fractal space-time [7,8], also independently introduced by El Naschie [8-15] is the geometric tool adapted to construct a new theory. We use here the word 'fractal' in its general meaning [16], denoting a set that shows structures at all scales and is thus explicitly resolution-dependent. More precisely, one can demonstrate [16] that the D_τ – measure of a continuous, almost everywhere non-differentiable set of topological dimension D_τ is a function of resolution, $L=L(\varepsilon)$, and diverges when resolution tends to zero, $L(\varepsilon)\to\infty$ when $\varepsilon\to 0$. In such a framework, resolutions are considered to be inherent to the description of the new, fractal, space-time. A new physical content may also be given to the concept of particles in this theory; various properties of 'particles' can be reduced to the geometric structures of the (fractal) geodesics of such a space-time [7].

Three levels of such a theory have been considered: (i) a 'Galileian' version corresponding to the standard fractals with constant fractal dimensions, and where dilation laws are the usual ones [7,8]. This theory provides us a new foundation of quantum mechanics from first principles; (ii) a special scale-relativistic version that implements in a more general way the principle of scale-relativity. It yields new dilatation laws of a Lorentzian form, that imply to re-interpret the Planck length-scale as a lower, impassable scale, invariant under dilatations [7,8]. The predictions of such a theory depart from that of standard quantum mechanics at large energies [7-15,17,18]; (iii) the third level, 'general scale-relativistic' version of the theory deals with non-linear scale laws and accounts for the coupling between scale laws and motion laws [7]. It yields a new interpretation of gauge invariance and allows one to get new mass-charge relations that solve the scale-hierarchy problem [7]. Using this theory [7], both conceptual (the complex nature of wave function, the probabilistic nature of quantum theory, the principle of correspondence, the quantum-classical transition, the divergence of masses and charges, the nature of Planck scale, the nature and quantization of electric charge, the origin of mass discretization of elementary particles, the nature of cosmological constant, etc.) and quantized results (the mass-charge relations, the electro-weak scale, the electron scale, the elementary fermion mass spectrum etc.) are obtained.

In the present paper the fractal model of atom is obtained in a generalized hydrodynamic formulation of SRT. Thus, in paragraph 2 we build a mathematical model that finally gives a generalized Navier-Stokes type equation and from here, for a special case of the movement, a generalized Schrödinger type equation, respectively the generalized hydrodynamic model. In Paragraph 3 this model is further applied to study the atom.

* *email: m.agop@yahoo.com*

Theoretical part

Mathematical model

Let us suppose that the electron motion around the nucleus takes place on fractal curves (continuous but non-differentiable curves) of fractal dimension D_F [19]. A manifold compatible with such movements defines a fractal space-time. The fractal nature of space-time implies, through non-differentiability, the breaking of differential time reflection invariance. In such a context, the usual definitions of the derivative of a given function with respect to time [7]:

$$\frac{df}{dt}=\lim_{\Delta t\to 0}\frac{f(t+\Delta t)-f(t)}{\Delta t}=\lim_{\Delta t\to 0}\frac{f(t)-f(t-\Delta t)}{\Delta t} \quad (1)$$

are equivalent in the differentiable case. One passes from one to the other by the transformation $\Delta t \to -\Delta T$ (time reflection invariance at the infinitesimal level). In the non-differentiable case two functions (df_+ / dt) and (df_- / dt) are defined as explicit functions of t and dt:

$$\frac{df_+}{dt}=\lim_{\Delta t\to 0}\frac{f(t+\Delta t,\Delta t)-f(t,\Delta t)}{\Delta t},$$
$$\frac{df_-}{dt}=\lim_{\Delta t\to 0}\frac{f(t,\Delta t)-f(t-\Delta t,\Delta t)}{\Delta t} \quad (2a,b)$$

The sign (+) corresponds to the forward process and (-) to the backward process.

Then, in the spaces coordinates d**X**, we can write [7]:

$$d\boldsymbol{X}_\pm = d\boldsymbol{x}_\pm + d\zeta_\pm = \boldsymbol{v}_\pm dt + d\zeta_\pm \quad (3a,b)$$

with $v_\pm$ the forward and backward mean speeds,

$$\boldsymbol{v}_+=\frac{d\boldsymbol{x}_+}{dt}=\lim_{\Delta t\to 0+}\left\langle\frac{\boldsymbol{X}(t+\Delta t)-\boldsymbol{X}(t)}{\Delta t}\right\rangle,$$
$$\boldsymbol{v}_-=\frac{d\boldsymbol{x}_-}{dt}=\lim_{\Delta t\to 0-}\left\langle\frac{\boldsymbol{X}(t)-\boldsymbol{X}(t-\Delta t)}{\Delta t}\right\rangle \quad (4a,b)$$

and $d\zeta_\pm$ a measure of non-differentiability (a fluctuation induced by the fractal properties of trajectory) having the average:

$$\langle d\xi_\pm\rangle = 0 \quad (5)$$

While the speed - concept is classically a single concept, if space-time is a fractal, we must introduce two speeds (v_+ and v_-) instead of one. This "two-valueness" of the speed vector is a new, specific consequence of non-differentiability that has no standard counterpart (in the sense of differential physics).

However, we cannot favor v_+ rather than v_-. The only solution is to consider both the forward (dt >0) and backward (dt <0) processes together. Than, it is necessary to introduce complex speed [7]:

$$\boldsymbol{V}=\frac{\boldsymbol{v}_+ + \boldsymbol{v}_-}{2}-i\frac{\boldsymbol{v}_+ - \boldsymbol{v}_-}{2}=\frac{d\boldsymbol{x}_+ + d\boldsymbol{x}_-}{2dt}-i\frac{d\boldsymbol{x}_+ - d\boldsymbol{x}_-}{2dt} \quad (6)$$

If $(v_+ + v_-)/2$ may be considered as differentiable (classical) speed, then the difference $(v_+ - v_-)/2$ is the non-differentiable (fractal) speed.

Using the notations $d\mathbf{x}_\pm = d_\pm \mathbf{x}$ equation (6) becomes:

$$\boldsymbol{V}=\left(\frac{d_+ + d_-}{2dt}-i\frac{d_+ - d_-}{2dt}\right)\boldsymbol{x} \quad (7)$$

This enables us to define the operator:

$$\frac{\delta}{dt}=\frac{d_+ + d_-}{2dt}-i\frac{d_+ - d_-}{2dt} \quad (8)$$

Let us now assume that the fractal curve is immersed in a 3-dimensional space, and that $\boldsymbol{X}$ of components X^i ($i=\overline{1,3}$) is the position vector of a point on the curve. Let us also consider a function $f(\mathbf{X},t)$ and the following Taylor series expansion up to the second order:

$$df=f(X^i+dX^i,t+dt)-f(X^i,dt)=$$
$$=\left(\frac{\partial}{\partial X^i}dX^i+\frac{\partial}{\partial t}dt\right)f(X^i,t)+\frac{1}{2}\left(\frac{\partial}{\partial X^i}dX^i+\frac{\partial}{\partial t}dt\right)^2 f(X^i,t) \quad (9)$$

From here, the forward and backward average values of this relation using notations $dX_\pm^i = dX_\pm^i$ take the form:

$$\langle d_\pm f\rangle=\left\langle\frac{\partial f}{\partial t}dt\right\rangle+\langle\nabla f\cdot d_\pm \boldsymbol{X}\rangle+\frac{1}{2}\left\langle\frac{\partial^2 f}{\partial t^2}(dt)^2\right\rangle+$$
$$+\left\langle\frac{\partial^2 f}{\partial X^i\partial t}d_\pm X^i dt\right\rangle+\frac{1}{2}\left\langle\frac{\partial^2 f}{\partial X^i\partial X^l}d_\pm X^i d_\pm X^l\right\rangle \quad (10)$$

We make the following stipulations: the mean values of the function f and its derivates coincide with themselves, and the differentials $dX_\pm^i$ and dt are independent, therefore the averages of their products coincide with the product of average. Thus equation (10) becomes:

$$d_\pm f=\frac{\partial f}{\partial t}dt+\nabla f\langle d_\pm \boldsymbol{X}\rangle+\frac{1}{2}\frac{\partial^2 f}{\partial t^2}\langle(dt)^2\rangle+$$
$$+\frac{\partial^2 f}{\partial X^i\partial t}\langle d_\pm X^i dt\rangle+\frac{1}{2}\frac{\partial^2 f}{\partial X^i\partial X^l}\langle d_\pm X^i d_\pm X^l\rangle \quad (11)$$

or more, using (3a,b),

$$d_\pm f=\frac{\partial f}{\partial t}dt+\nabla f d_\pm \boldsymbol{x}+\frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2+\frac{\partial^2 f}{\partial X^i\partial t}d_\pm x^i dt+$$
$$+\frac{1}{2}\frac{\partial^2 f}{\partial X^i\partial X^l}\left(d_\pm x^i d_\pm x^l+\langle d\xi_\pm^i d\xi_\pm^l\rangle\right) \quad (12)$$

Since $d\zeta_\pm^i$ describes the fractal properties of the trajectory with the fractal dimension D_F [7,16], it is natural to impose $(d\zeta_\pm^i)^{DF}$ to be proportional with dt, *i.e.*

$$(d\xi_\pm^i)^{D_F}=\sqrt{2D}dt \quad (13)$$

where D is a coefficient of proportionality.

Let us focus now on the mean $\langle d\zeta_\pm^i d\zeta_\pm^l\rangle$. If $i \neq l$ this average is zero due the independence of $d\zeta^i$ and $d\zeta^l$. So, using (13) we can write:

$$\langle d\xi_\pm^i d\xi_\pm^l\rangle=\pm\delta^{il}2D(dt)^{(2/D_F)-1} \quad (14)$$

with

$$\delta^{il}=\begin{cases}1, & if \quad i=l\\ 0, & if \quad i\neq l\end{cases}$$

and we had considered that:

$$\begin{cases}\langle d\xi_+^i d\xi_+^l\rangle>0 \ and \ dt>0\\ \langle d\xi_-^i d\xi_-^l\rangle>0 \ and \ dt<0\end{cases}$$

Then (12) may be written under the form:

$$d_\pm f=\frac{\partial f}{\partial t}dt+\nabla f d_\pm \boldsymbol{x}+\frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2+\frac{\partial^2 f}{\partial X^i\partial t}d_\pm x^i dt+ +\frac{1}{2}\frac{\partial^2 f}{\partial X^i\partial X^l}d_\pm x^i d_\pm x^l \pm\frac{\partial^2 f}{\partial X^i \partial X^l}\delta^{il}D(dt)^{(2/D_F)-1} \quad (15)$$

If we divide it by d*t* and neglect the terms which contain differential factors, (15) is reduced to:

$$\frac{d_\pm f}{dt}=\frac{\partial f}{\partial t}+\boldsymbol{v}_\pm\nabla f_\pm \pm D(dt)^{(2/D_F)-1}\Delta f \quad (16)$$

Under the circumstances, let us calculate, $\delta f\ /\ dt$ According with (8) and taking into account (16), we have:

$$\frac{\delta f}{dt}=\frac{1}{2}\left[\frac{d_+f}{dt}+\frac{d_-f}{dt}-i\left(\frac{d_+f}{dt}-\frac{d_-f}{dt}\right)\right]=$$

$$=\frac{1}{2}\left[\left(\frac{\partial f}{\partial t}+\boldsymbol{v}_+\nabla f+D(dt)^{(2/D_F)-1}\Delta f\right)+\left(\frac{\partial f}{\partial t}+\boldsymbol{v}_-\nabla f-D(dt)^{(2/D_F)-1}\Delta f\right)\right]-$$

$$-\frac{i}{2}\left[\left(\frac{\partial f}{\partial t}+\boldsymbol{v}_+\nabla f+D(dt)^{(2/D_F)-1}\Delta f\right)-\left(\frac{\partial f}{\partial t}+\boldsymbol{v}_-\nabla f-D(dt)^{(2/D_F)-1}\Delta f\right)\right]=$$

$$=\frac{\partial f}{\partial t}+\left(\frac{\boldsymbol{v}_++\boldsymbol{v}_-}{2}-i\frac{\boldsymbol{v}_+-\boldsymbol{v}_-}{2}\right)\nabla f-iD(dt)^{(2/D_F)-1}\Delta f \quad (17)$$

or using (6):

$$\frac{\delta f}{dt}=\frac{\partial f}{\partial t}+\boldsymbol{V}\cdot\nabla f-iD(dt)^{(2/D_F)-1}\Delta f\ , \quad (18)$$

This relation also allows us to give the definition of the fractal operator:

$$\frac{\delta}{dt}=\frac{\partial}{\partial t}+\boldsymbol{V}\cdot\nabla-iD(dt)^{(2/D_F)-1}\Delta \quad (19)$$

We now apply the principle of scale covariance, and postulate that the passage from classical (differentiable) mechanics to the “fractal” (non-differentiable) mechanics that is considered here can be implemented by replacing the standard time derivative *d/dt* by the new complex operator δ/dt (this results in a generalization of the principle of scale covariance given by Nottale in [7]). As a consequence, we are now able to write the equation of geodesics (a generalization of the first Newton’s principle) in a fractal space-time under its covariant form:

$$\frac{\delta \boldsymbol{V}}{dt}=\frac{\partial \boldsymbol{V}}{\partial t}+\boldsymbol{V}\cdot\nabla\boldsymbol{V}-iD(dt)^{(2/D_F)-1}\Delta\boldsymbol{V}=0 \quad (20)$$

i.e. a generalized Navier-Stokes type equation with an imaginary viscosity coefficient $\eta=iD(dt)^{(2/DF)-1}$ This means that the complex global acceleration field, $\delta V/dt$ depends on the complex local acceleration field, $\delta_t V$, on the non-linear (convective) term, $V.\nabla V$ and on the dissipative one, ΔV. Moreover, the behavior of a “fractal fluid” is of viscoelastic or of hysteretic type [20-22].

From equation (20), using the operational relation, $V.\nabla V=\nabla(V^2/2)-V.(\nabla.V)$, we obtain,

$$\frac{\delta \boldsymbol{V}}{dt}=\frac{\partial \boldsymbol{V}}{\partial t}+\nabla\left(\frac{\boldsymbol{V}^2}{2}\right)-\boldsymbol{V}\times(\nabla\times\boldsymbol{V})-iD(dt)^{(2/D_F)-1}\Delta\boldsymbol{V}=0 \quad (21)$$

If the movement of the “fractal fluid” is irrotational, *i.e.* $\Omega=\nabla.V=0$, we can choose **V** of the form:

$$\boldsymbol{V}=\nabla\phi \quad (22)$$

with ϕ a complex speed potential. Then (21) becomes:

$$\frac{\delta \boldsymbol{V}}{dt}=\frac{\partial \boldsymbol{V}}{\partial t}+\nabla\left(\frac{\boldsymbol{V}^2}{2}\right)-iD(dt)^{(2/D_F)-1}\nabla^2\boldsymbol{V}=0 \quad (23)$$

and more, by substituting equation (22) in equation (23) we shall have by integration:

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}(\nabla\phi)^2-iD(dt)^{(2/D_F)-1}\Delta\phi=F(t) \quad (24)$$

with F(t) a function of time. We observe that equation (23) has been reduced to a single scalar relation (24), *i.e.* a generalized Bernoulli-type equation.

Let us choose the complex speed potential in the form:

$$\phi=-2iD(dt)^{(2/D_F)-1}\ln\psi \quad (25a)$$

By means of equation (24) the function ψ satisfies a generalized Schrödinger type equation,

$$D^2(dt)^{(4/D_F)-2}\Delta\psi+D(dt)^{(2/D_F)-1}\partial_t\psi=F(t) \quad (25b)$$

Moreover, for $D=\hbar/2m_0$ with $\hbar$ the reduced Planck constant and m_0 the rest mass of a test particle and the fractal dimension $D_F=2$ (*e.g.* Peano type curves which completely cover a two-dimensional surface – see Nottale’s approach of the SRT [7]), equation (25b) for $F(t)=0$ is reduced to the usual Schrödinger equation. Then ψ simultaneously behaves as speed potential and wave function.

For $\psi=\sqrt{\rho}e^{iS}$, with $\sqrt{\rho}$ the amplitude and *S* the phase of ψ, the complex speed ***V*** in the form **V**= ν + *iu* has the components:

$$\boldsymbol{v}=2D(dt)^{(2/D_F)-1}\nabla S,\ \boldsymbol{u}=-D(dt)^{(2/D_F)-1}\nabla\ln\rho \quad (26a,b)$$

By substituting the complex speed field ***V*** of components (26a,b) in equation (23) and separating the real and imaginary parts, we obtain:

$$m_0\left(\frac{\partial\boldsymbol{v}}{\partial t}+\boldsymbol{v}\cdot\nabla\boldsymbol{v}\right)=-\nabla(Q)$$
$$\frac{\partial\boldsymbol{u}}{\partial t}+\nabla\left(\boldsymbol{v}\cdot\boldsymbol{u}+D(dt)^{(2/D_F)-1}\nabla\cdot\boldsymbol{v}\right)=0 \quad (27a, b)$$

with *Q* the fractal potential,

$$Q=-2m_0D(dt)^{(2/D_F)-1}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}=$$
$$-\frac{m_0\boldsymbol{u}^2}{2}-m_0D(dt)^{(2/D_F)-1}\nabla\cdot\boldsymbol{u} \quad (28)$$

Equation (27b), by integration up to an arbitrary phase factor which may be set to zero by a suitable choice of the phase ψ, corresponds to the conservation law of the probability density:

$$\partial_t\rho+\nabla\cdot(\rho\boldsymbol{v})=0 \quad (29)$$

In a scalar field *U*, equation (27a) takes the form

$$m_0\left(\frac{\partial\boldsymbol{v}}{\partial t}+\boldsymbol{v}\cdot\nabla\boldsymbol{v}\right)=-\nabla(Q+U) \quad (30)$$

and corresponds to the momentum conservation law. Equations (29) and (30) form the equations system of hydrodynamics in the fractal space-time.

The wave function of ψ(r,t) is invariant when its phase changes by an integer multiple of 2π. Indeed, equation (26a) gives:

$$\oint m_0 \boldsymbol{v} d\boldsymbol{r} = 2m_0 D(dt)^{(2/D_F)-1} \oint dS =$$
$$= 4\pi n m_0 D(dt)^{(2/D_F)-1}, \quad n = 0, \pm 1, \pm 2, ... \qquad (31)$$

a condition of compatibility between the SRT hydrodynamic model and the wave mechanics.

Particularly, for $D=\hbar/2m$ and $D_F = 2$ (30) takes the standard form

$$\oint \boldsymbol{p}\cdot d\boldsymbol{r} = nh$$

The set of equations (29) and (30) represents a complete system of differential equations for the fields ρ(r,t) and ν(r,t); relation (31) relates each solution $(\rho,\nu)_n$ with the wave solution ψ in a unique way.

The field ρ(r,t) is a probability distribution, namely the probability of finding the particle in the vicinity d**r** of the point r at time *t*,

$$dP = \rho d\boldsymbol{r}, \quad \iiint \rho d\boldsymbol{r} = 1, \qquad (32a,b)$$

the space integral being extended over the entire area of the system. Any time variation of the probability density ρ(r,t) is accompanied by a probability current ρν pointing towards or outwards, the corresponding field point **r** (29).

The position probability of the real velocity field ν(r,t) (30), varies with space and time similar to a hydrodynamic fluid placed in the force-field of an external potential and U(**r**,t)a fractal potential (28). The fluid (in the sense of a statistical particles ensemble) exhibits, however, an essential difference compared to an ordinary fluid: in a rotation motion ν(r,t) increases (decreases) with the distance from the center **r** decreasing (increasing) (31).

The expectation values for the real velocity field and the velocity operator $\hat{\boldsymbol{v}} = -2iD(dt)^{(2/D_F)-1}\nabla$ of wave mechanics are equal,

$$\langle \boldsymbol{v} \rangle = \iiint \rho \boldsymbol{v} d\boldsymbol{r} = \iiint \Psi^* \hat{\boldsymbol{v}} \Psi d\boldsymbol{r} = \langle \hat{\boldsymbol{v}} \rangle_{WM} \qquad (33)$$

but in the higher-order, |*n*| > 2, similar identities are invalid, namely $\langle \nu^{in} \rangle \neq \langle \nu^n \rangle$. The expectation for the 'fractal force' vanishes at all times (theorem of Ehrenfest [6]), *i.e.*

$$\langle -\nabla Q \rangle = \iiint \rho(-\nabla Q) d\boldsymbol{r} = 0 \qquad (34)$$

or explicitly

$$2m_0 D^2 (dt)^{(4/D_F)-2} \iiint \rho \nabla \left(\frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}} \right) d\boldsymbol{r} =$$
$$= m_0 D^2 (dt)^{(4/D_F)-2} \oint (\rho \nabla\nabla \ln \rho) \cdot d\boldsymbol{\sigma} = 0 \qquad (35)$$

Two types of fractal stationary states are distinguished:

i) *Dynamic states*. For ∂/∂*t* = 0 and , ν≠0 equations (30) and (29) give:

$$\nabla\left(\frac{1}{2} m_0 \boldsymbol{v}^2 + U + Q \right) = 0, \qquad \nabla(\rho \boldsymbol{v}) = 0 \qquad (36a,b)$$

namely

$$\frac{1}{2} m_0 \boldsymbol{v}^2 + U + Q = E, \qquad \rho \boldsymbol{v} = \nabla \times \boldsymbol{F} \qquad (37a,b)$$

Consequently, inertia $m_0\nu \cdot \nabla\nu$, exterior forces (-∇*U*), and fractal forces (-∇*Q*) are in balance at every field point (36a). The sum of the kinetic energy $m_0\nu^2/2$, external (*U*) and fractal potential energy (*Q*) is invariant, *i.e.* equal to the integration constant *E*≠ *E*(*r*) (37a). *E* ≡ <*E*> represents the total energy of the dynamic system. The probability flow density ρν has no sources (36b), *i.e.* its streamlines are closed (37b) .

ii) *Static states*. For ∂/∂t= 0 and ν=0, equations (30) and (29) give:

$$\nabla(U + Q) = 0 \qquad (38)$$

i.e.

$$U + Q = E \qquad (39)$$

The exterior force (-∇*U*) is balanced by the fractal force (-∇*Q*) at any field point (38). The sum of the exterior (*U*) and interior (*Q*) potential energy is invariant, *i.e.* equal to the integration constant, *E* ≠ *E*(*r*) eq. (29) is identically satisfied. *E* ≡ <*E*> represents the total energy of the fractal static system.

Results and discussions

Hydrodynamic model of the fractal atom

Let us consider an electron orbiting in the electric field of the nucleus, $U=-e^2/(4\pi\varepsilon_0 r)$. The collective chaotic effect on considered electron of all the other electrons has as result a motion on a fractal curve. For example, if $D=\hbar/2m_0$, and $D_F = 2$, a Brownian-like motion can be chosen [23]. A particularization of equation (37a) for the case of stationary motion gives:

$$\nabla^2 \sqrt{\rho} = -\frac{1}{2m_0 D^2 (dt)^{(4/D_F)-2}} \left(E - \frac{1}{2} m_0 \boldsymbol{v}^2 + \frac{e^2}{4\pi\varepsilon_0 r} \right) \sqrt{\rho}, \quad \nabla(\rho \boldsymbol{v}) = 0$$
$$0 < r < \infty, \quad 0 \le \varphi \le \pi, \quad 0 \le \Phi \le 2\pi,$$
$$\rho(r = \infty, \varphi, \Phi) = 0, \quad \rho(r, \varphi, \Phi)_{\varphi \le \pi/2} = \rho(r, \varphi + \pi/2, \Phi)$$
$$(40a-g)$$

The probability currents can flow only in closed lines. Because of the azimuthal symmetry of the system, the speed potential is of the form $S=2mm_0D(dt)^{(2/DF)-1}$ eq. (26a) with (32a,b), corresponding to a speed field independent of Φ

$$\boldsymbol{v} = \frac{2mD(dt)^{(2/D_F)-1} \boldsymbol{e}_\Phi}{r \sin\varphi}, \quad m = 0, \pm 1, \pm 2, ... \qquad (41a,b)$$

This statement is in agreement with the compatibility condition, (31),

$$\oint m_0 \boldsymbol{v} \cdot d\boldsymbol{r} = 2mm_0 (dt)^{(2/D_F)-1} \oint \frac{\boldsymbol{e}_\Phi d\boldsymbol{r}}{r \sin\varphi} =$$
$$= 2mm_0 (dt)^{(2/D_F)-1} \int_0^{2\pi} d\Phi = 4\pi m m_0 (dt)^{(2/D_F)-1} \qquad (42)$$

By the substitution of relation (41a), equation (40a) reduces to the equation:

$$\nabla^2 \sqrt{\rho} = -\frac{1}{2m_0 D^2 (dt)^{(4/D_F)-2}} \left(E - \frac{2m_0 m^2 D^2 (dt)^{(4/D_F)-2}}{r^2 \sin^2\varphi} + \frac{e^2}{4\pi\varepsilon_0 r} \right) \sqrt{\rho} \qquad (43)$$

which is independent of Φ. In other words, ρ = ρ(*r*, φ*)* since $\nu || e_\Phi$ and (40b) is satisfied. By means of the statement,

$$\sqrt{\rho} = R(r)\Pi(\varphi) \qquad (44)$$

Equation (43) can be separated into two differential equations with respect to *r* and φ, respectively,

(λ - separation parameter)

$$\lambda = -\frac{1}{\Pi} \left[\frac{1}{\sin\varphi} \frac{d}{d\varphi} \left(\sin\varphi \frac{d\Pi}{d\varphi} \right) - \frac{m^2}{\sin^2\varphi} \Pi \right]$$
$$\frac{r^2}{R} \left[\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \frac{1}{2m_0 D^2 (dt)^{(4/D_F)-2}} \left(E + \frac{e^2}{4\pi\varepsilon_0 r} \right) R \right] = \lambda \qquad (45a, b)$$

Equations (45a,b) have solution if the constants *E* and λ assume the eigenvalues [6]:

$$\lambda_\ell = \ell(\ell + 1), \quad \ell = 0, 1, 2, ... \qquad (46)$$

and

$$E_m = -\frac{m_0}{2m^2}\left(\frac{e^2}{8\pi\varepsilon_0 m_0 D(dt)^{(2/D_F)-1}}\right)^2, \quad m = 1, 2, 3, ... \qquad (47)$$

The solutions of equation (45b) are $R(r) = \xi^{\ell}\, exp(-\xi/2) L_{n+\ell}^{2l+1}(\xi)$ where $\zeta = 2r/na_0$ and

$$a_0 = 8D^2 (dt)^{(4/D_F)-2}\left(\frac{4\pi\varepsilon_0 m_0}{e^2}\right) \qquad (48)$$

and the solutions of equation (45b) are $\Pi(\varphi) \sim P_l^{\,m}(cos\,\varphi)$. Thus one finds after normalization that [6]

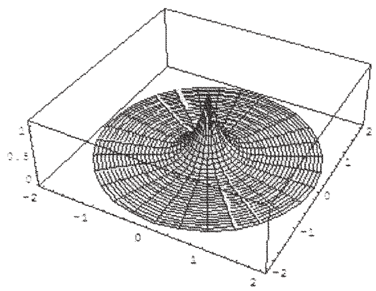
$$\rho_{\ell mn} = C_{\ell mn}\left(\frac{2r}{na_0}\right)^{2\ell} e^{-\frac{2r}{na_0}}\left[L_{n+\ell}^{2\ell+1}\left(\frac{2r}{na_0}\right) P_\ell^m(\cos\varphi)\right]^2$$

$$C_{\ell mn} = \left(\frac{2}{na_0}\right)^3 \frac{(n-\ell-1)!}{2n[(n+\ell)!]^3}\,\frac{2\ell+1}{4\pi}\,\frac{(\ell-|m|)!}{(\ell+|m|)!} \qquad (49a,b)$$

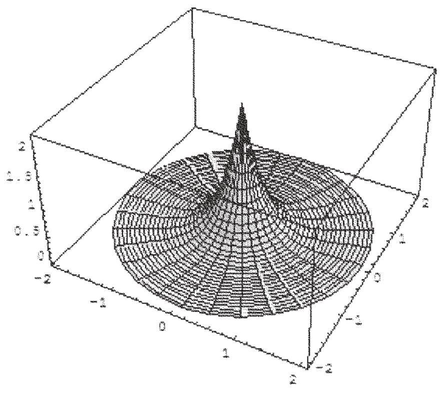
For physical reasons, only the following combinations of quantum numbers are acceptable:

$$0 \le \ell \le n-1 \qquad -\ell \le m \le +\ell \qquad (50)$$

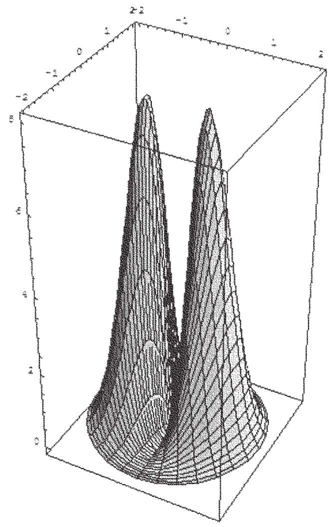
Relations (41a,b) and (49a,b) represent the complete solution $(\rho,v)_{min}$ of the SRT hydrodynamic model of the fractal atom. In figure 1, the probability density dependence on the polar normalized coordinates, for various quantum numbers, is graphically represented. The ρ_{lmn} probability density behavior for n=1,2 and different values of the *(l,m)* pairs, *[a(l=0;m=0;n=1), b(l=0;m=0;n=2), c(l=1; m=0;n=2), d(l=1;m=1;n=2), e(l=1;m=-1;n=2])*, is shown below.



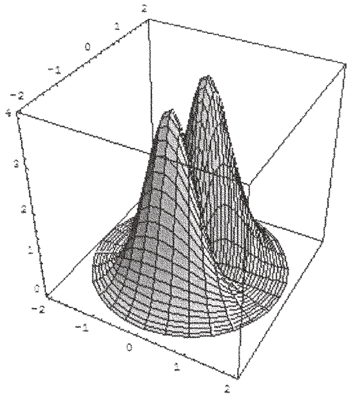
a) $\rho_{001} \times a_0^3$



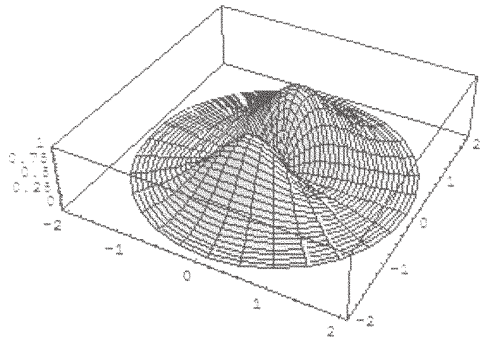
b) $\rho_{002} \times a_0^3 \times 10^2$



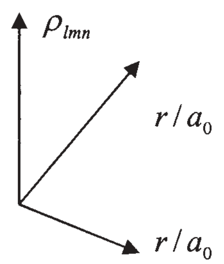
c) $\rho_{102} \times a_0^3 \times 10^3$



d) $\rho_{112} \times a_0^3 \times 10^3$



e) $\rho_{1-12} \times a_0^3 \times 10^3$



f) The coordinate axes

Fig. 1. The probability density dependence on the polar normalized coordinates, for various quantum numbers, in 3D representation

By means of the recurrence relations for the associated Laguerre and Legendre polynomials [4], one shows from the solution (49a,b) that equation (37a) becomes

$$Q = -2m_0 D^2 (dt)^{(4/D_F)-2} \left(\frac{1}{n^2 a_0^2} - \frac{2}{a_0 r} + \frac{m^2}{r^2 \sin^2 \varphi} \right) \qquad (51)$$

while

$$U(r) = -4m_0 D^2 (dt)^{(4/D_F)-2} \frac{1}{a_0 r} \qquad (52)$$

and

$$\frac{1}{2} m_0 v^2 = \frac{2m_0 m^2 D^2 (dt)^{(4/D_F)-2}}{r^2 \sin^2 \varphi} \qquad (53)$$

It can be seen that the fractal potential energy (Q) overcompensates the electric energy (U) and the kinetic energy $m_0 v^2/2$ at any field point (r, φ, Φ). The remaining energy is finite and represents the observable energy of the system:

$$\frac{1}{2} m_0 v^2 + U + Q = -2m_0 D^2 (dt)^{(4/D_F)-2} \frac{1}{a_0^2 n^2} \equiv E \qquad (54)$$

The states with $m = 0$ are static states (ν=0) and the states with $m \neq 0$ are dynamic states (ν≠0) eq. (41a,b), (51) and (53)). In any state with $m \neq 0$, the rotation motion decreases with increasing distance r, *i.e.* for a given direction φ, $\nu_\Phi \sim 1/r$ (41a,b).

Conclusions

The main conclusions of the present paper are as follows:

i) a generalization of the Nottale's scale relativity theory is given. The generalized Schrödinger equation is obtained as an irrotational movement of generalized Navier-Stokes type fluids having an imaginary viscosity coefficient. Then, ψ simultaneously becomes wave-function and speed potential;

ii) a hydrodynamic model of the scale relativity theory is built;

iii) one can stress out that the quantum potential introduced in the hydrodynamic model of quantum mechanics comes from the non-differentiability of the fractal space-time;

iv) applying this model to study the atom, it resulted that the real part of the complex velocity field describes the electron averaged movement. In this case the electron moves on stationary orbits according to a quantification condition. The imaginary part of the complex velocity describes the fractality through a fractal potential. Now, using this potential, from the averaged movements (on stationary orbits), the electron energy quantification results;

v) in the D_F=2 fractal dimension and for the $D = \hbar / 2m$ viscosity coefficient, the classical results of quantum mechanics for the hydrogen atom are obtained.

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